

# Qubit Entanglement Driven by Remote Optical Fields

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We examine the entanglement between two qubits, supposed to be remotely located and driven by independent quantized optical fields. No interaction is allowed between the qubits, but their degree of entanglement changes as a function of time. We report a collapse and revival of entanglement that is similar to the collapse and revival of single-atom properties in cavity QED.

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Control of the evolution of qubit entanglement and the time-dependent behavior of qubit pairs in networks is relevant for quantum computing and cryptography. Entanglement will be stored in quantum memory registers (see conceptual sketch in Fig. 1) for eventual use in some form of quantum communication. The qubits must be controlled in some way externally, and we are interested here in the response of the entanglement of a pair of stored qubits to quantized optical control fields.

What we can call “pure storage” requires mutual isolation and non-interaction between qubits. Two-qubit evolution has been studied without qubit isolation and usually allowing or relying on mutual interactions to produce entanglement dynamics. Kim, et al., showed [1] that an incoherent thermal field can create entanglement between two such qubits. Entanglement transfer between two qubits and two separate fields was examined by Zhou and Wang [2]. In their treatment the quantum field was weak rather than strong. The evolution of entanglement in a qubit-field system, where the qubit and the field start from mixed states was examined by Rendell and Rajagopal [3]. They aimed to calculate the entanglement embedded in the full system, and because of the lack of an entanglement measure for  $2 \times \infty$  systems they calculated a lower bound for the concurrence instead.

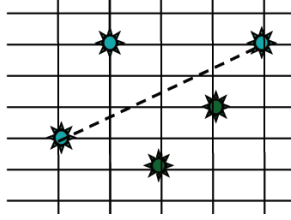


Fig. 1. Sketch indicating non-interacting qubits in a quantum storage network. Dashed line indicates two are entangled.

Other studies [4, 5] have shown that two isolated qubits can exhibit periodic fluctuations in their entanglement in the form of early-stage decoherence (ESD - also referred to as entanglement sudden death) [6] when the qubits are modelled as “controlled” locally by interaction with only single photons. However, manageable control fields are better modelled as containing many photons. Here we retain a quantum picture of two many-photon well-phased control fields by using a coherent state description of them with a large mean photon number  $\bar{n} \gg 1$ .

For our calculations we take single-mode control fields. Each field is assumed, for simplicity, to be exactly resonant with the flip transition between the ground  $|g\rangle$  and excited  $|e\rangle$  states of the qubit that it addresses. The well-known Jaynes-Cummings (JC) interaction [7] is then relevant at each qubit site (labelled  $i = 1, 2$ ), and the interaction Hamiltonian is given by:

$$H_I = \sum_{i=1,2} \hbar g (a_i \sigma_i^+ + a_i^\dagger \sigma_i^-), \quad (1)$$

where  $a_i$  and  $a_i^\dagger$  are the photon annihilation and creation operators for site  $i$ , and  $\sigma_i^+$  and  $\sigma_i^-$  are the raising and lowering Pauli matrices for atom  $i$ , and  $g$  is the coupling constant between atoms and fields, taken the same for both for greatest simplicity hereafter. For the two coherent state fields we take the same  $\bar{n}$  for simplicity, and we assume initial entanglement in the form of a familiar Bell State:

$$|\Psi(0)\rangle = (|eg\rangle + |ge\rangle)/\sqrt{2}. \quad (2)$$

Since JC time evolution is unitary, no information can be truly lost in evolution via  $H_I$ , but the infinite range of photon numbers in a coherent state brings aspects of open-system theory into play. However, the quasi-classical nature of practically available control fields suggests that we not expect their quantum characteristics to be dominant. Thus we will trace out the fields and follow only the qubit entanglements.

In this discussion we will use Wootters’ concurrence [9] as our entanglement measure,

which is given by

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (3)$$

where the quantities  $\lambda_i$  are the eigenvalues in decreasing order of the matrix

$$\zeta = \rho(\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y). \quad (4)$$

Here  $\rho$  is our two-qubit reduced-state density matrix,  $\rho^*$  denotes the complex conjugation of  $\rho$  in the standard basis, and  $\sigma_y$  is the Pauli matrix expressed in the same basis.

The photon number in a coherent field is Poisson distributed and relatively tightly centered around  $\bar{n}$  when  $\bar{n} \gg 1$ . This suggests a shortcut approximation, to be checked numerically, in which we represent the field density matrix as a Fock state having photon number equal to  $\bar{n}$ . An important simplification occurs in taking the same  $\bar{n}$  for both control fields. Then the initial field state is  $|\bar{n}\rangle \otimes |\bar{n}\rangle$  and the reduced density matrix for the qubits is:

$$\rho = \begin{pmatrix} a & x & x & x \\ x & b & z & x \\ x & z^* & c & x \\ x & x & x & d \end{pmatrix} \rightarrow \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \quad (5)$$

where we have used the standard two-qubit basis  $[ee, eg, ge, gg]$ . The elements indicated by  $x$  are zero because of the equal- $\bar{n}$  simplification. Thus, under the assumptions mentioned,  $\rho$  is of  $X$ -type (see [8]).

For an  $X$ -type  $\rho$ , Eq.(3) turns into:

$$C(\rho) = 2 \max[0, |z| - \sqrt{ad}]. \quad (6)$$

The control fields induce growth in time of the elements  $a$ ,  $d$ , which are the only ones not already present in the original maximally entangled state (2). Their growth and any decline of  $z$  will cause entanglement to decrease.

Having used the Fock state shortcut to obtain (5), we avoid using it further now and calculate the elements  $z$ ,  $a$ ,  $d$  for the coherent state. We introduce the Poisson number distribution by the coherent-state amplitude measure  $A_n = e^{-|\alpha|^2/2} \alpha^n / \sqrt{n!}$ , where  $|\alpha|^2 = \bar{n}$ , we obtain a doubly infinite series summation:

$$\begin{aligned} z &= \frac{1}{2} \left\{ \sum_{n,m} A_n^2 A_m^2 C_n C_{n+1} C_m C_{m+1} \right. \\ &\quad - A_n A_{n-1} A_m A_{m+1} S_n C_{n+1} C_m S_{m+1} \\ &\quad + A_n A_{n-2} A_m A_{m+2} S_n S_{n-1} S_{m+1} S_{m+2} \\ &\quad \left. - A_n A_{n-1} A_m A_{m+1} S_n C_{n-1} S_{m+1} C_{m+2} \right\}, \end{aligned} \quad (7)$$

where  $C_n = \cos(gt\sqrt{n})$  and  $S_n = \sin(gt\sqrt{n})$ .

Similarly the series summations for  $a$  and  $d$  are;

$$\begin{aligned}
a = & \frac{1}{2} \left\{ \sum_{n,m} A_n^2 A_m^2 C_{n+1}^2 S_m^2 \right. \\
& + A_n A_{n+1} A_m A_{m-1} S_{n+1} C_{n+1} S_m C_m \\
& + A_n^2 A_m^2 S_n^2 C_{m+1}^2 \\
& \left. + A_n A_{n-1} A_m A_{m+1} S_n C_n S_{m+1} C_{m+1} \right\}
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
d = & \frac{1}{2} \left\{ \sum_{n,m} A_n^2 A_m^2 S_{n+1}^2 C_m^2 \right. \\
& + A_n A_{n+1} A_m A_{m-1} S_{n+1} C_{n+1} S_m C_m \\
& + A_n^2 A_m^2 C_n^2 S_{m+1}^2 \\
& \left. + A_n A_{n-1} A_m A_{m+1} S_n C_n S_{m+1} C_{m+1} \right\}.
\end{aligned} \tag{9}$$

The infinite extent of these summations of course reflects the fact that we have coupled the qubits to an infinite state space. The sums cannot be analytically completed, but in our calculations we choose  $\alpha = 10$ , i.e.,  $\bar{n} = 100 \gg 1$ , so we can obtain good approximations if we use Stirling's formula for  $n!$ ,

$$n! = \sqrt{2\pi n} n^n e^{-n}, \tag{10}$$

and Euler's formula to approximate the terms in the summations above by integrals. If we approximate the terms like  $A_n A_{n+1} A_m A_{m-1}$  with  $A_n^2 A_m^2$ , which introduces an error of order  $1/\bar{n}$  near the Poisson peaks  $n \approx m \approx \bar{n}$ , helpful cancellations can be identified, and we obtain an approximate expression for  $|z| - \sqrt{ad}$ :

$$\begin{aligned}
|z| - \sqrt{ad} \cong & \frac{1}{4} \left[ e^{-g^2 t^2 / 8\bar{n}^2} - 1 \right] \\
& + \frac{1}{2} \left[ \sum_n A_n^2 \cos(2gt\sqrt{n}) \right]^2 \\
& - \frac{1}{2} \left[ \sum_n A_n^2 \sin(2gt\sqrt{n}) \right]^2.
\end{aligned} \tag{11}$$

The summations in (11) involving  $\cos(2gt\sqrt{n})$  and  $\sin(2gt\sqrt{n})$  also cannot be completed analytically, but are the same type as those for qubit inversion in the original discussion of quantum revivals [10] for zero detuning, so we expect to see similar revival behavior here. Fig. 2 shows our analytical results for the function  $2(|z| - \sqrt{ad})$  in comparison with the complete numerical evaluation of Eq.(3) without making the  $X$ -type approximation.

We note that both curves in Fig. 2 indicate repeated occurrences of early-stage decoherence (ESD). The recurring positive entanglement events are not periodic in amplitude, as found in

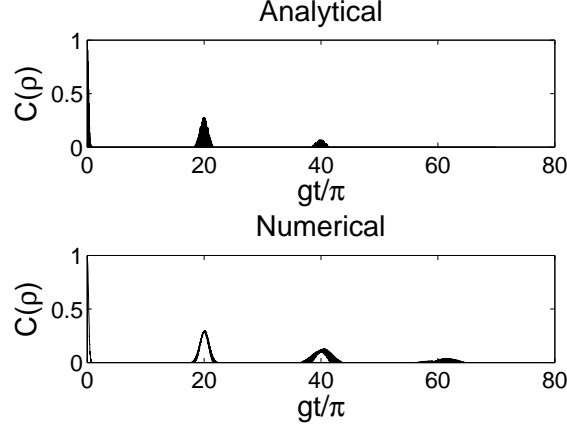


Fig. 2. The analytical and numerical results for entanglement. As expected, revivals occur, which are predicted by the approximate analytical results reasonably well. Better resolution of the rapid oscillations is provided in Fig. 3.

the weak-field cases discussed in [4, 5], but are periodic in time. Fig. 3 below shows that the analytic approximation includes micro-ESD events that are not present in the full expression.

Because of the exponential term in (11) the envelope of the function decreases slowly. Also, close to the revival regions the function makes rapid oscillations with period  $\tau = \pi/(2g\sqrt{\bar{n}})$  which is half the corresponding period in the inversion case. The reason is that in (11) we have squares of summations rather than the summations for the original inversion calculations [10]. As the figure shows, our analytic summations and the approximations they are based on work rather well, capturing all major aspects of the full numerical result. The function  $2(|z| - \sqrt{ad})$  can suitably predict the magnitude and revival of the entanglement while not a perfect substitute for it.

In conclusion, we have made a numerical and analytic examination of the entanglement dynamics between two qubits controlled by (or, receiving “instructions” from) quantized optical fields that are modelled as equal-amplitude coherent state modes. We show that periodic ESD effects, as were predicted for very weak control fields in [4, 5], are still present but in substantially modified form. Many very rapid appearances of the ESD effect are contained in the top plot of Fig. 3 and are not present in the numerical evaluation of concurrence for the complete two-qubit reduced  $\rho$ . Both the X-state simplification in format and the high- $\bar{n}$  approximation contribute to the differences, but the latter is much less significant when  $\bar{n} \gg 1$ , as here. A visible consequence in both plots coming just from the open-system nature of the coherent states is the imperfect recovery of entanglement in each successive revival zone.

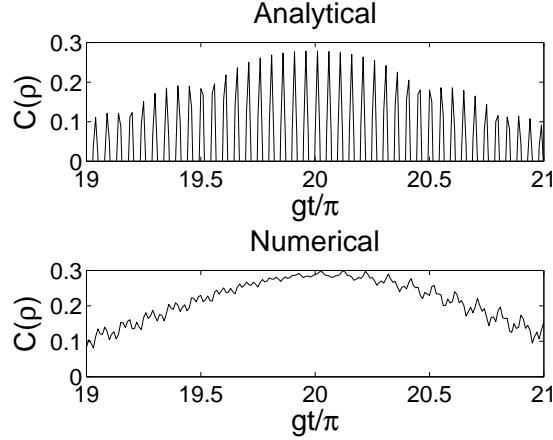


Fig. 3. A more detailed plot of the results around  $t = 20\pi/g$  shown in Fig. 2. Analytical results are for the  $X$ -type  $\rho$  while the numerical ones are for the original  $\rho$ .

Figs. 2 and 3 have implications for further work. They show that the major features of the entangled qubits' response are very well captured even by the rather severe  $X$ -state simplification we introduced. The benefits of having analytic expressions, as provided by the  $X$ -state format, can be expected to be substantial in guiding and interpreting more complex calculations.

In this regard, we believe it will be interesting to investigate dynamic entanglement behavior under different assumptions about the state of the system. For example, the response to squeezed fields or fields acting at different times or fields of significantly different intensity or mode frequency are open for study. Similarly, the state reduction employed to reach the  $X$  state is not limited to two-qubit situations. These expanded topics will be the focus of a wider investigation.

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## References

1. M. S. Kim, J. Lee, D. Ahn and P. L. Knight, "Entanglement induced by a single-mode heat environment", Phys. Rev. A **65**, 040101(R) (2002).
2. L. Zhou and G. Yang, "Entanglement reciprocation between atomic qubits and entangled coherent state", J. Phys. B. **39**, 5143-5150 (2006).
3. R. W. Rendell and A. K. Rajagopal, "Revivals and entanglement from initially entangled mixed states of a damped Jaynes-Cummings model", Phys. Rev. A **67**, 062110 (2003).
4. M. Yönaç, T. Yu and J. H. Eberly, "Sudden death of entanglement of two Jaynes-

- Cummings atoms”, J. Phys. B. **39**, S621-S625 (2006).
5. M. Yönaç, T. Yu and J. H. Eberly, “Pairwise concurrence dynamics: a four-qubit model”, J. Phys. B. **40**, S45 (2007).
  6. T. Yu and J. H. Eberly, “ Finite-Time Disentanglement Via Spontaneous Emission”, Phys. Rev. Lett. **93**, 140404 (2004).
  7. E.T. Jaynes and F.W. Cummings, “Comparison of quantum and semiclassical radiation theories with application to the beam maser”, Proc. IEEE **51**, 89 (1963).
  8. T. Yu and J.H. Eberly, “Evolution from Entanglement to Decoherence”, Quant. Inf. and Comp., **7**, 459 (2007).
  9. W. K. Wootters, “Entanglement of Formation of an Arbitrary State of Two Qubits”, Phys. Rev. Lett. **80**, 2245 (1998).
  10. J. H. Eberly, N. B. Narozhny and J. J. Sanchez Mondragon, “Periodic spontaneous collapse and revival in a simple quantum model”, Phys. Rev. Lett. **44**, 20 (1980).

## References

1. M. S. Kim, J. Lee, D. Ahn and P. L. Knight, Phys. Rev. A **65**, 040101(R) (2002).
2. L. Zhou and G. Yang, J. Phys. B. **39**, 5143-5150 (2006).
3. R. W. Rendell and A. K. Rajagopal, Phys. Rev. A **67**, 062110 (2003).
4. M. Yönaç, T. Yu and J. H. Eberly, J. Phys. B. **39**, S621-S625 (2006).
5. M. Yönaç, T. Yu and J. H. Eberly, J. Phys. B. **40**, S45 (2007).
6. T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
7. E.T. Jaynes and F.W. Cummings, Proc. IEEE **51**, 89 (1963).
8. T. Yu and J.H. Eberly, Quant. Inf. and Comp., **7**, 459 (2007).
9. W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
10. J.H. Eberly, N.B. Narozhny and J.J. Sanchez Mondragon, Phys. Rev. Lett. **44**, 20 (1980).